

General Disclaimer

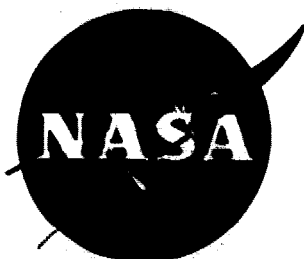
One or more of the Following Statements may affect this Document

- This document has been reproduced from the best copy furnished by the organizational source. It is being released in the interest of making available as much information as possible.
- This document may contain data, which exceeds the sheet parameters. It was furnished in this condition by the organizational source and is the best copy available.
- This document may contain tone-on-tone or color graphs, charts and/or pictures, which have been reproduced in black and white.
- This document is paginated as submitted by the original source.
- Portions of this document are not fully legible due to the historical nature of some of the material. However, it is the best reproduction available from the original submission.

NASA TECHNICAL MEMORANDUM

JSC-09379

**NASA TM X-58182
May 1976**



**THE ADAMS FORMULAS FOR NUMERICAL INTEGRATION
OF DIFFERENTIAL EQUATIONS FROM 1ST TO 20TH ORDER**

(NASA-TM-X-58182) THE ADAMS FORMULAS FOR
NUMERICAL INTEGRATION OF DIFFERENTIAL
EQUATIONS FROM 1ST TO 20TH ORDER (NASA)

37 p HC \$4.00

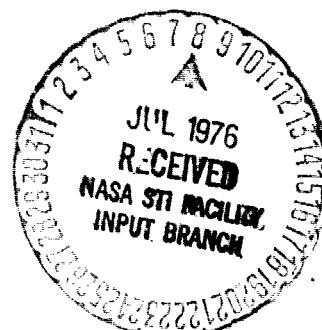
CSCI 12A

N76-26919

**Unclas
42323**

G3/64

**NATIONAL AERONAUTICS AND SPACE ADMINISTRATION
LYNDON B. JOHNSON SPACE CENTER
HOUSTON, TEXAS 77058**



1. Report No. NASA TM X-58182		2. Government Accession No.		3. Recipient's Catalog No.	
4. Title and Subtitle THE ADAMS FORMULAS FOR NUMERICAL INTEGRATION OF DIFFERENTIAL EQUATIONS FROM 1ST TO 20TH ORDER				5. Report Date May 1976	
				6. Performing Organization Code JSC-09379	
7. Author(s) James C. Kirkpatrick				8. Performing Organization Report No.	
9. Performing Organization Name and Address Lyndon B. Johnson Space Center Houston, Texas 77058				10. Work Unit No. 986-16-00-00-72	
				11. Contract or Grant No.	
12. Sponsoring Agency Name and Address National Aeronautics and Space Administration Washington, D.C. 20546				13. Type of Report and Period Covered Technical Memorandum	
				14. Sponsoring Agency Code	
15. Supplementary Notes					
16. Abstract The Adams-Bashforth predictor coefficients and the Adams-Moulton corrector coefficients for the integration of differential equations are presented for methods of 1st to 20th order. The order of the method as presented in this report refers to the highest order difference formula used in Newton's backward difference interpolation formula, on which the Adams method is based. The Adams method is a polynomial approximation method derived from Newton's backward difference interpolation formula. The Newton formula is derived and expanded to 20th order. The Adams predictor and corrector formulas are derived and expressed in terms of differences of the derivatives, as well as in terms of the derivatives themselves. All coefficients are given to 18 significant digits. For the difference formula only, the ratio coefficients are given to 10th order.					
17. Key Words (Suggested by Author(s)) Predictors Runge-Kutta method Formulas (mathematics) Interpolation Coefficients Extrapolation Polynomials Differential equations Measure and integration				18. Distribution Statement STAR Subject Category: 64 (Numerical Analysis)	
19. Security Classif. (of this report) Unclassified		20. Security Classif. (of this page) Unclassified		21. No. of Pages 37	
				22. Price* \$4.00	

*For sale by the National Technical Information Service, Springfield, Virginia 22161

THE ADAMS FORMULAS FOR NUMERICAL INTEGRATION
OF DIFFERENTIAL EQUATIONS FROM 1ST TO 20TH ORDER

James C. Kirkpatrick
Lyndon B. Johnson Space Center
Houston, Texas 77058

CONTENTS

Section	Page
SUMMARY	1
INTRODUCTION	1
ANALYSIS	2
CONCLUDING REMARKS	9
REFERENCE	9
APPENDIX A — NEWTON'S BACKWARD DIFFERENCE FORMULA EXPANDED TO 20TH ORDER	A-1
APPENDIX B — ADAMS PREDICTOR AND CORRECTOR FORMULAS FOR 1ST- TO 20TH-ORDER SYSTEMS IN TERMS OF DERIVATIVE VALUES	B-1
APPENDIX C — ADAMS PREDICTOR AND CORRECTOR DIFFERENCE FORMULAS	C-1
APPENDIX D — EXPANSION OF DIFFERENCE FORMULAS IN TERMS OF TABULATED VALUES	D-1

PRECEDING PAGE BLANK NOT FILMED

THE ADAMS FORMULAS FOR NUMERICAL INTEGRATION OF DIFFERENTIAL EQUATIONS FROM 1ST TO 20TH ORDER

By James C. Kirkpatrick
Lyndon B. Johnson Space Center

SUMMARY

The Adams integration method is a fast, accurate polynomial approximation method for integrating differential equations numerically. The method, which was derived from Newton's backward difference interpolation formula, requires only two derivative evaluations for each integration step. The formulas are derived in terms of differences of the derivatives, as well as in terms of the derivatives themselves. All coefficients are given to 18 significant digits. For the difference formula only, the ratio coefficients are given to 10th order. The binomial and factorial coefficients associated with the difference formulas are also given to 20th order.

The accuracy of the Adams integration method is a function of (1) the accuracy of the first $n + 1$ derivative values, which must be initially supplied, and (2) the order of the method chosen in relationship to the word length of the computer. (The higher the order of the method used, the smaller the size of the integration step that can be used; however, an order that is too high or too low can lead to serious errors and cause the solution to diverge.)

INTRODUCTION

The Adams formulas constitute one of the fastest and most accurate methods for numerical integration of differential equations; however, there are two major drawbacks: the formulas are not self-starting, and the amount of labor associated with the computation of the higher order formulas is nearly prohibitive. The first drawback can be overcome by using a self-starting method to initiate the solution. Of the self-starting methods, the fourth-order Runge-Kutta method (with the Gill modification) and the Fehlberg 7/8 method are the most common. The second drawback can be overcome by judicious programming. It is hoped that the results presented in this report will eliminate the need for programming the solution for the computation of required coefficients.

The Adams method is a polynomial approximation method derived from Newton's backward difference interpolation formula. The Newton formula is derived in the analysis section, and the results are expanded to 20th order in appendix A. The Adams formulas are also derived in the analysis section, and those results are expanded for use in 1st- to 20th-order systems in appendix B. Appendixes C and D provide, respectively, the expansions of the Newton formula used in the derivation

of the Adams formulas and the difference formulas in terms of the tabulated function values. Both expansions are carried out to 20th order.

ANALYSIS

Newton's backward difference formula is used for interpolating a value of a function $y = f(x)$ near the end of an ordered set of tabular values y_i . Thus, if $y = f(x)$ denotes a function that takes the values $y_n, y_{n-1}, y_{n-2}, \dots, y_1$ for the $n+1$ equidistant values $x_n, x_{n-1}, x_{n-2}, \dots, x_1$ of the independent variable x , then finding a suitable polynomial $F(x)$ to replace the tabulated function over a given interval can be accomplished (ref. 1) by

$$\begin{aligned}
 F(x) &= a_0 + \sum_{i=0}^{n-1} a_{i+1} \prod_{k=n-i}^n (x - x_k) \\
 &= a_0 + a_1(x - x_n) + a_2(x - x_{n-1})(x - x_n) \\
 &\quad + a_3(x - x_{n-2})(x - x_{n-1})(x - x_n) \\
 &\quad \dots \dots \dots \\
 &\quad + a_r(x - x_{n-r+1})(x - x_{n-r+2}) \dots (x - x_n) \\
 &\quad \dots \dots \dots \\
 &\quad + a_n(x - x_1)(x - x_2) \dots (x - x_n)
 \end{aligned} \tag{1}$$

The coefficients $a_0, a_1, a_2, \dots, a_n$ will be determined to make $F(x_n) = y_n$, $F(x_{n-1}) = y_{n-1}$, $F(x_{n-2}) = y_{n-2}$, \dots , $F(x_1) = y_1$.

Substituting successive values x_n, x_{n-1}, \dots, x_1 for x , together with the corresponding tabular function values y_n, y_{n-1}, \dots, y_1 , in equation (1) gives

$$y_n = a_0 \quad \text{or} \quad a_0 = y_n \tag{2}$$

$$\begin{aligned}
y_{n-1} &= a_0 + a_1(x_{n-1} - x_n) \\
&= y_n + a_1(x_{n-1} - x_n)
\end{aligned} \tag{3}$$

Because the x_i are equidistant, $x_{i+1} - x_i = h$ and

$$a_1 = \frac{(y_n - y_{n-1})}{h} = \frac{\Delta_1 y_n}{h} \tag{4}$$

Similarly,

$$\begin{aligned}
y_{n-2} &= a_0 + a_1(x_{n-2} - x_n) + a_2(x_{n-2} - x_{n-1})(x_{n-1} - x_n) \\
&= y_n + \frac{y_n - y_{n-1}}{h}(-2h) + a_2(-h)(-2h) \\
&= -y_n + 2y_{n-1} + 2a_2h^2 \\
a_2 &= \frac{y_n - 2y_{n-1} + y_{n-2}}{2h^2} = \frac{\Delta_2 y_n}{2h^2}
\end{aligned} \tag{5}$$

Equation (5) may be verified by recalling that

$$\begin{aligned}
\Delta_2 y_n &= \Delta_1 y_n - \Delta_1 y_{n-1} \\
&= (y_n - y_{n-1}) - (y_{n-1} - y_{n-2}) \\
&= y_n - 2y_{n-1} + y_{n-2}
\end{aligned} \tag{6}$$

A list of the higher order differences in terms of the tabulated function values is given in appendix D.

$$\begin{aligned}
y_{n-3} &= a_0 + a_1(x_{n-3} - x_n) + a_2(x_{n-3} - x_{n-1})(x_{n-3} - x_n) \\
&\quad + a_3(x_{n-3} - x_{n-2})(x_{n-3} - x_{n-1})(x_{n-3} - x_n) \\
&= y_n + \frac{y_n - y_{n-1}}{h}(-3h) + \frac{y_n - 2y_{n-1} + y_{n-2}}{2h^2}(-2h)(-3h) \\
&\quad + a_3(-h)(-2h)(-3h) \\
&= y_n(1 - 3 + 3) + y_{n-1}(3 - 6) + y_{n-2}(3) - 6h^3 a_3 \\
&= y_n - 3y_{n-1} + 3y_{n-2} - 6h^3 a_3 \\
a_3 &= \frac{y_n - 3y_{n-1} + 3y_{n-2} - y_{n-3}}{6h^3} = \frac{\Delta_3 y_n}{3!h^3}
\end{aligned} \tag{7}$$

In general, for $1 \leq i \leq n$,

$$a_i = \frac{\Delta_i y_n}{i!h^i} \tag{8}$$

Equation (1) can now be written as follows.

$$\begin{aligned}
F(x) &= y_n + \sum_{i=0}^{n-1} \frac{\Delta_{i+1} y_n}{(i+1)!} \prod_{k=n-i}^n \left(\frac{x - x_k}{h} \right) \\
&= y_n + \Delta_1 y \left(\frac{x - x_n}{h} \right) + \frac{\Delta_2 y_n}{2} \left(\frac{x - x_{n-1}}{h} \right) \left(\frac{x - x_n}{h} \right) \\
&\quad \dots \\
&\quad + \frac{\Delta_r y_n}{r!} \left(\frac{x - x_{n-r+1}}{h} \right) \left(\frac{x - x_{n-r+2}}{h} \right) \dots \left(\frac{x - x_n}{h} \right) \\
&\quad \dots \\
&\quad + \frac{\Delta_n y_n}{n!} \left(\frac{x - x_1}{h} \right) \left(\frac{x - x_2}{h} \right) \dots \left(\frac{x - x_n}{h} \right)
\end{aligned} \tag{9}$$

Let

$$u = \frac{x - x_n}{h} \quad \text{or} \quad x = x_n + hu \quad (10)$$

Then, because $x_{n-1} = x_n - h$, $x_{n-2} = x_n - 2h$, . . . , the above factors can be written as follows.

$$\left. \begin{aligned} \frac{x - x_{n-1}}{h} &= \frac{x - (x_n - h)}{h} = \frac{x - x_n}{h} + \frac{h}{h} = u + 1 \\ \frac{x - x_{n-2}}{h} &= \frac{x - (x_n - 2h)}{h} = \frac{x - x_n}{h} + \frac{2h}{h} = u + 2 \\ &\dots\dots\dots \\ \frac{x - x_1}{h} &= \frac{x - x_{n-(n-1)}}{h} = u + n - 1 \end{aligned} \right\} \quad (11)$$

Substituting equation (11) in equation (9) gives

$$\begin{aligned} F(x) &= y_n + \sum_{k=1}^n \frac{\Delta_k y_n}{k!} \prod_{i=0}^{k-1} (u + i) \\ &= y_n + u \Delta_1 y_n + \frac{u(u+1)}{2!} \Delta_2 y_n \\ &\quad + \frac{u(u+1)(u+2)}{3!} \Delta_3 y_n + \dots \\ &\quad + \frac{u(u+1)(u+2) \dots (u+n-1)}{n!} \Delta_n y_n \end{aligned} \quad (12)$$

Equation (12) is Newton's backward difference formula. It has been expanded to 20th order in appendix C. The formula can be placed in terms of the function values if the difference formulas in appendix D are used.

If the first $n + 1$ derivative values of the derivatives of a differential equation are known at $n + 1$ consecutive, equally spaced values of the independent variable, then a polynomial can be fitted to these data and the equation can be integrated by integrating the polynomial. (The appropriate polynomial for this purpose is that given by equation (12)). Integration can be done because, in the numerical integration of differential equations, the solution is always begun from a set of given values

(initial conditions) and constructed from that point forward. Hence, the values of the function immediately behind at a given point of the solution are always known, but the values ahead are unknown. The problem of finding the next value ahead is solved by the Adams predictor and corrector formulas, which are derived as follows.

By applying Newton's formula to a set of ordered values of the derivatives of a function to be integrated (e.g., writing $y' = dy/dx$ for y in equation (12) and using the expanded terms from appendix C),

$$y' = y'_n + u \Delta_1 y'_n + (u^2 + u) \frac{\Delta_2 y'_n}{2} + (u^3 + 3u^2 + 2u) \frac{\Delta_3 y'_n}{6} + (u^4 + 6u^3 + 11u^2 + 6u) \frac{\Delta_4 y'_n}{24} + \dots \quad (13)$$

Because the change in y for any interval is given by the formula

$$\Delta y = \int_{x_k}^{x_{k+1}} \left(\frac{dy}{dx} \right) dx = \int_{x_k}^{x_{k+1}} y' dx \quad (14)$$

equation (13) can be used to compute the change in y over any interval wherein dy/dx is continuous. Therefore, over any interval $x_{k+1} - x_k$,

$$\Delta y = \int_{x_k}^{x_{k+1}} \left[y'_n + u \Delta_1 y'_n + (u^2 + u) \frac{\Delta_2 y'_n}{2} + (u^3 + 3u^2 + 2u) \frac{\Delta_3 y'_n}{6} + (u^4 + 6u^3 + 11u^2 + 6u) \frac{\Delta_4 y'_n}{24} + \dots \right] dx \quad (15)$$

Because

$$\left. \begin{aligned} x &= x_n + hu \\ dx &= h du \end{aligned} \right\} \quad (16)$$

equation (15) becomes

$$\Delta y = h \left[u y'_n + \frac{u^2}{2} \Delta_1 y'_n + \left(\frac{u^3}{3} + \frac{u^2}{2} \right) \frac{\Delta_2 y'_n}{2} \right. \\ \left. + \left(\frac{u^4}{4} + u^3 + u^2 \right) \frac{\Delta_3 y'_n}{6} \right. \\ \left. + \left(\frac{u^5}{5} + \frac{3u^4}{2} + \frac{11u^3}{3} + 3u^2 \right) + \dots \right]_{u_k}^{u_{k+1}} \quad (17)$$

after integration with the change in limits $x_{k+1} \rightarrow u_{k+1}$ and $x_k \rightarrow u_k$. In the interval $x_{n+1} - x_n$,

$$\left. \begin{aligned} u_{k+1} &= \frac{x_{n+1} - x_n}{h} = \frac{h}{h} = 1 \\ u_k &= \frac{x_n - x_n}{h} = \frac{0}{h} = 0 \end{aligned} \right\} \quad (18)$$

Integrating equation (17) between the limits 0 and 1 gives the following predictor formula.

$$\Delta y = \int_n^{n+1} = h \left[y'_n + \frac{1}{2} \Delta_1 y'_n + \frac{5}{12} \Delta_2 y'_n + \frac{3}{8} \Delta_3 y'_n + \frac{251}{720} \Delta_4 y'_n \right. \\ \left. + \frac{95}{288} \Delta_5 y'_n + \frac{19087}{60480} \Delta_6 y'_n + \frac{5257}{17280} \Delta_7 y'_n + \frac{1070017}{3628800} \Delta_8 y'_n \right. \\ \left. + \frac{25713}{89600} \Delta_9 y'_n + \frac{26842253}{95800320} \Delta_{10} y'_n + \dots \right] \quad (19)$$

In the interval $x_{n-1} - x_n$,

$$\left. \begin{aligned} u_{k+1} &= \frac{x_n - x_n}{h} = \frac{0}{h} = 0 \\ u_k &= \frac{x_{n-1} - x_n}{h} = \frac{-h}{h} = -1 \end{aligned} \right\} \quad (20)$$

Integrating equation (17) between the limits -1 and 0 gives the following corrector formula.

$$\begin{aligned} \Delta y = \int_{n-1}^n = h & \left[y'_n - \frac{1}{2} \Delta_1 y'_n - \frac{1}{12} \Delta_2 y'_n - \frac{1}{24} \Delta_3 y'_n - \frac{19}{720} \Delta_4 y'_n \right. \\ & - \frac{3}{160} \Delta_5 y'_n - \frac{863}{80480} \Delta_6 y'_n - \frac{275}{24192} \Delta_7 y'_n - \frac{33953}{3628800} \Delta_8 y'_n \\ & \left. - \frac{8183}{1036800} \Delta_9 y'_n - \frac{32504331}{479001600} \Delta_{10} y'_n - \dots \right] \end{aligned} \quad (21)$$

Equations (19) and (21) are the predictor and corrector formulas of the Adams method and are to be used as follows. Use equation (19) to compute a predicted value,

$$\bar{y}_{n+1} = y_n + \int_n^{n+1} \quad (22)$$

Place a bar over y_{n+1} to indicate that it is a first approximation obtained by extrapolation. Use this value of y_{n+1} to compute the value of the derivative y'_{n+1} . With the value of the y'_{n+1} derivative thus obtained, compute a corrected value of y_{n+1} by using equation (21) with the n and $n-1$ replaced by k and $k-1$, wherein $k = n + 1$.

$$\begin{aligned} y_{n+1} &= y_n + \int_{k-1}^k \\ &= y_n + h \left[y'_{n+1} - \frac{1}{2} \Delta_1 y'_{n+1} - \frac{1}{12} \Delta_2 y'_{n+1} - \frac{1}{24} \Delta_3 y'_{n+1} - \dots \right] \end{aligned} \quad (23)$$

Equations (22) and (23) have been placed in an easy-to-use form in appendix B. The equations given in appendix B are written in terms of the derivative values at the various intervals ranging from a 1st- to 20th-order system. For an n th-order system, $n + 1$ derivative values are required.

CONCLUDING REMARKS

The Adams integration method is a fast, accurate method for integrating differential equations numerically. It requires only two derivative evaluations for each integration step, but additional time and storage must be dedicated to continually updating or reordering the table of past derivative values. When large sets of differential equations are integrated by using a high-order method, the amount of time required to update the derivative table can become appreciable. When this occurs, the Adams formula is preferred for its speed.

The accuracy of the Adams integration method is a function of (1) the accuracy of the first $n + 1$ derivative values, which must be initially supplied, and (2) the order of the method chosen in relation to the word length of the computer. The higher the order of the method used, the smaller the size of the integration step that can be used; however, an order that is too high or too low can lead to serious errors and cause the solution to diverge. For single-precision work, no order higher than seventh or eighth should be used with a nine-digit word length. For double-precision work, 13th is probably the highest order that should be attempted with a 20-digit word length. A 10th-order method can be used successfully with a 12-digit word length, and an order up to 20th can be used successfully with a 24-digit word length.

Because the data in this report were computed using floating-point arithmetic, only the first 16 digits of each 18-digit number are recommended without reservation.

Lyndon B. Johnson Space Center
National Aeronautics and Space Administration
Houston, Texas, April 20, 1976
986-16-00-00-72

REFERENCE

1. Scarborough, J. B.: Numerical Mathematical Analysis. The Johns Hopkins Press, 1966, pp. 59-60 and 320-322.

APPENDIX A

NEWTON'S BACKWARD DIFFERENCE FORMULA EXPANDED TO 20TH ORDER

The expansion of Newton's backward difference formula to 20th order is as follows.

$$\begin{aligned}
 y' = y'_n + \sum_{k=1}^n \frac{\Delta_k y'_n}{k!} \prod_{i=0}^{k-1} (u + i) = y'_n + u \Delta_1 y'_n \\
 + (u^2 + u) \frac{\Delta_2 y'_n}{2} \\
 + (u^3 + 3u^2 + 2u) \frac{\Delta_3 y'_n}{6} \\
 + (u^4 + 6u^3 + 11u^2 + 6u) \frac{\Delta_4 y'_n}{24} \\
 + (u^5 + 10u^4 + 35u^3 + 50u^2 + 24u) \frac{\Delta_5 y'_n}{120} \\
 + (u^6 + 15u^5 + 85u^4 + 225u^3 + 274u^2 + 120u) \frac{\Delta_6 y'_n}{720} \\
 + (u^7 + 21u^6 + 175u^5 + 735u^4 + 1624u^3 + 1764u^2 + 720u) \frac{\Delta_7 y'_n}{5040} \\
 + (u^8 + 28u^7 + 322u^6 + 1960u^5 + 6769u^4 + 13132u^3 + 13068u^2 + 5040u) \frac{\Delta_8 y'_n}{40320} \\
 + (u^9 + 36u^8 + 546u^7 + 4536u^6 + 22449u^5 + 67284u^4 + 118124u^3 + 109584u^2 \\
 + 40320u) \frac{\Delta_9 y'_n}{362880} \\
 + (u^{10} + 45u^9 + 870u^8 + 9450u^7 + 63273u^6 + 269325u^5 + 723680u^4 + 1172700u^3 \\
 + 1026576u^2 + 362880u) \frac{\Delta_{10} y'_n}{3628800}
 \end{aligned}$$

$$\begin{aligned}
& + (u^{11} + 55u^{10} + 1320u^9 + 18150u^8 + 157773u^7 + 902055u^6 + 3416930u^5 + 8409500u^4 \\
& + 127535760u^3 + 10628640u^2 + 3628800u) \frac{\Delta_{11}y'_n}{39916800} \\
& + (u^{12} + 66u^{11} + 1925u^{10} + 32670u^9 + 357423u^8 + 2637558u^7 + 13339535u^6 \\
& + 45995730u^5 + 105258076u^4 + 150917965u^3 + 120543840u^2 + 39916800u) \frac{\Delta_{12}y'_n}{479001600} \\
& + (u^{13} + 78u^{12} + 2717u^{11} + 55770u^{10} + 749463u^9 + 6926634u^8 + 44990231u^7 \\
& + 206070150u^6 + 657206836u^5 + 1414014888u^4 + 1931559552u^3 + 1486442880u^2 \\
& + 475091600u) \frac{\Delta_{13}y'_n}{6227020800} \\
& + (u^{14} + 91u^{13} + 3731u^{12} + 91091u^{11} + 1474473u^{10} + 16669653u^9 + 135036473u^8 \\
& + 790943153u^7 + 3336118786u^6 + 9957703756u^5 + 20313753096u^4 + 26596717056u^3 \\
& + 19802759040u^2 + 6227020800u) \frac{\Delta_{14}y'_n}{87178291200} \\
& + (u^{15} + 105u^{14} + 5005u^{13} + 143325u^{12} + 2749747u^{11} + 37312275u^{10} + 368411615u^9 \\
& + 2681453775u^8 + 14409322928u^7 + 56663366760u^6 + 159721605680u^5 + 310989260400u^4 \\
& + 392156797824u^3 + 283465647360u^2 + 87178291200u) \frac{\Delta_{15}y'_n}{1307674368000} \\
& + (u^{16} + 120u^{15} + 6580u^{14} + 218400u^{13} + 4899622u^{12} + 78558480u^{11} + 928095740u^{10} \\
& + 8207628000u^9 + 54631129553u^8 + 272803210680u^7 + 1009672107080u^6 \\
& + 2706813345600u^5 + 5056995703824u^4 + 6165817614720u^3 + 4339163001600u^2 \\
& + 1307674368000u) \frac{\Delta_{16}y'_n}{20922789888000}
\end{aligned}$$

$$\begin{aligned}
& + (u^{17} + 136u^{16} + 8500u^{15} + 323680u^{14} + 8394022u^{13} + 156952432u^{12} + 2185031420u^{11} \\
& + 23057159840u^{10} + 185953177553u^9 + 1146901283528u^8 + 5374523477960u^7 \\
& + 18861567058880u^6 + 48366009233424u^5 + 87077748875904u^4 + 102992244837120u^3 \\
& + 70734282393600u^2 + 20922789888000u) \frac{\Delta_{17}y'_n}{355687428096000} \\
& + (u^{18} + 153u^{17} + 10812u^{16} + 468180u^{15} + 13896582u^{14} + 299650806u^{13} + 4853222764u^{12} \\
& + 60202693980u^{11} + 577924894833u^{10} + 4308105301929u^9 + 24871845297936u^8 \\
& + 110228466184200u^7 + 369012649234384u^6 + 909299905844112u^5 + 1583313975727488u^4 \\
& + 1821602444624640u^3 + 1223405590579200u^2 + 355687428096000u) \frac{\Delta_{18}y'_n}{6402373705728000} \\
& + (u^{19} + 171u^{18} + 13566u^{17} + 662796u^{16} + 22323822u^{15} + 549789282u^{14} + 10246937272u^{13} \\
& + 147560703732u^{12} + 1661573386473u^{11} + 14710753408923u^{10} + 102417740732658u^9 \\
& + 557921681547048u^8 + 2353125040549984u^7 + 7551527592063024u^6 \\
& + 17950712280921504u^5 + 30321254007719424u^4 + 34012249593822720u^3 \\
& + 22376988058521600u^2 + 6402373705728000u) \frac{\Delta_{19}y'_n}{121645100408832000} \\
& + (u^{20} + 190u^{19} + 16815u^{18} + 920550u^{17} + 34916946u^{16} + 973941900u^{15} + 20692933630u^{14} \\
& + 342252511900u^{13} + 4465226757381u^{12} + 46280647751910u^{11} + 381922055502195u^{10} \\
& + 2503858755467550u^9 + 12953636989943896u^8 + 52260903362512720u^7 \\
& + 161429736530118960u^6 + 371384787345228000u^5 + 610116075740491775u^4 \\
& + 668609730341153279u^3 + 431565146817638400u^2 \\
& + 121645100408832000u) \frac{\Delta_{20}y'_n}{2432902008176640000}
\end{aligned}$$

APPENDIX B

ADAMS PREDICTOR AND CORRECTOR FORMULAS FOR 1ST- TO 20TH-ORDER SYSTEMS IN TERMS OF DERIVATIVE VALUES

The predictor and corrector methods using derivative values directly are as follows.

1ST-ORDER SYSTEM

Predictor

$$y_{n+1} = y_n + \Delta T(-0.5y'_{n-1} + 1.5y'_n)$$

Corrector

$$y_{n+1} = y_n + \Delta T(0.5y'_n + 0.5y'_{n+1})$$

2ND-ORDER SYSTEM

Predictor

$$y_{n+1} = y_n + \Delta T(0.4166666666666667y'_{n-2} - 0.1333333333333333 + 001y'_{n-1} \\ + 0.1916666666666667 + 001y'_n)$$

Corrector

$$y_{n+1} = y_n + \Delta T(-0.8333333333333335 - 001y'_{n-1} + 0.6666666666666667y'_n \\ + 0.4166666666666668y'_{n+1})$$

3RD-ORDER SYSTEM

Predictor

$$y_{n+1} = y_n + \Delta T(-0.375y'_{n-3} + 0.1541666666666666 + 001y'_{n-2} \\ - 0.2458333333333333 + 001y'_{n-1} + 0.2291666666666666 + 001y'_n)$$

Corrector

$$y_{n+1} = y_n + \Delta T(0.416666666666665 - 001y'_{n-2} - 0.2083333333333333y'_{n-1} \\ + 0.7916666666666666y'_n + 0.375y'_{n+1})$$

4TH-ORDER SYSTEM

Predictor

$$y_{n+1} = y_n + \Delta T(0.3486111111111111y'_{n-4} - 0.1769444444444444 + 001y'_{n-3} \\ + 0.3633333333333333 + 001y'_{n-2} - 0.3852777777777777 + 001y'_{n-1} \\ + 0.2640277777777777 + 001y'_n)$$

Corrector

$$y_{n+1} = y_n + \Delta T(-0.26388888888888892 - 001y'_{n-3} + 0.1472222222222224y'_{n-2} \\ - 0.36666666666666669y'_{n-1} + 0.8972222222222223y'_n \\ + 0.34861111111111113y'_{n+1})$$

Predictor

$$\begin{aligned} \mathbf{y}_{n+1} = & \mathbf{y}_n + \Delta T \left(-0.32986111111111111111 \mathbf{y}'_{n-5} + 0.199791666666666666 + 001 \mathbf{y}'_{n-4} \right. \\ & - 0.506805555555555554 + 001 \mathbf{y}'_{n-3} + 0.693194444444444443 + 001 \mathbf{y}'_{n-2} \\ & \left. - 0.5502083333333333331 + 001 \mathbf{y}'_{n-1} + 0.297013888888888888 + 001 \mathbf{y}'_n \right) \end{aligned}$$

$$y_{n+1} = y_n + \Delta T(0.1875 - 0.001y'_{n-4} - 0.12013888888888889y'_{n-3} + 0.33472222222222223y'_{n-2} - 0.55416666666666668y'_{n-1} + 0.99097222222222222y'_n + 0.32986111111111114y'_{n+1})$$

Predictor

$$y_{n+1} = y_n + \Delta T (0.315591931216931216y'_{n-6} - 0.222341269841269840 + 001y'_{n-5} \\ + 0.673179563492063490 + 001y'_{n-4} - 0.113798941798941799 + 002y'_{n-3} \\ + 0.116658234126984127 + 002y'_{n-2} - 0.739563492063492062 + 001y'_{n-1} \\ + 0.328573082010582010 + 001y'_n)$$

$$y_{n+1} = y_n + \Delta T \left(-0.142691798941798941 - 0.01y'_{n-5} + 0.104365079365079364y'_{n-4} \right. \\ \left. - 0.334176587301587300y'_{n-3} + 0.620105820105820105y'_{n-2} \right. \\ \left. - 0.768204365079365079y'_{n-1} + 0.107658730158730159 + 0.01y'_n \right. \\ \left. + 0.315591931216931219y'_{n+1} \right)$$

7TH-ORDER SYSTEM

Predictor

$$\begin{aligned}
 y_{n+1} = y_n + \Delta T & (-0.304224537037037036y'_{n-7} + 0.244516369047619047 + 001y'_{n-6} \\
 & - 0.861212797619047617 + 001y'_{n-5} + 0.173796544312169311 + 002y'_{n-4} \\
 & - 0.220277529761904761 + 002y'_{n-3} + 0.180545386904761904 + 002y'_{n-2} \\
 & - 0.952520667989417986 + 001y'_{n-1} + 0.358995535714285713 + 001y'_n)
 \end{aligned}$$

Corrector

$$\begin{aligned}
 y_{n+1} = y_n + \Delta T & (0.113673941798941799 - 001y'_{n-6} - 0.938409391534391534 - 001y'_{n-5} \\
 & + 0.343080357142857143y'_{n-4} - 0.732035383597883596y'_{n-3} \\
 & + 0.101796461640211640 + 001y'_{n-2} - 0.100691964285714286 + 001y'_{n-1} \\
 & + 0.115615906084656084 + 001y'_n + 0.304224537037037040y'_{n+1})
 \end{aligned}$$

8TH-ORDER SYSTEM

Predictor

$$\begin{aligned}
 y_{n+1} = y_n + \Delta T & (0.294868000440917107y'_{n-8} - 0.266316854056437389 + 001y'_{n-7} \\
 & + 0.107014677028218694 + 002y'_{n-6} - 0.251247360008818341 + 002y'_{n-5} \\
 & + 0.380204144620811286 + 002y'_{n-4} - 0.385403610008818341 + 002y'_{n-3} \\
 & + 0.263108427028218694 + 002y'_{n-2} - 0.118841506834215167 + 002y'_{n-1} \\
 & + 0.388482335758377424 + 001y'_n)
 \end{aligned}$$

Corrector

$$\begin{aligned}
 y'_{n+1} = y_n + \Delta T & (-0.5653659611992939 - 002y'_{n-7} + 0.862196869488536150 - 001y'_{n-6} \\
 & - 0.355823963844797176y'_{n-5} + 0.867046406525573189y'_{n-4} \\
 & - 0.138699294532627865 + 001y'_{n-3} + 0.154193066578483245 + 001y'_{n-2} \\
 & - 0.126890266754850088 + 001y'_{n-1} + 0.123101135361552028 + 001y'_n \\
 & + 0.294868000440917110y'_{n+1})
 \end{aligned}$$

9TH-ORDER SYSTEM

Predictor

$$\begin{aligned}
 y_{n+1} = y_n + \Delta T & (-0.286975446428571428y'_{n-9} + 0.287764701829805996 + 001y'_{n-8} \\
 & - 0.129942846119929453 + 002y'_{n-7} + 0.348074052028218694 + 002y'_{n-6} \\
 & - 0.612836422508818339 + 002y'_{n-5} + 0.741793207120811284 + 002y'_{n-4} \\
 & - 0.626462985008818339 + 002y'_{n-3} + 0.366419587742504408 + 002y'_{n-2} \\
 & - 0.144669297012786595 + 002y'_{n-1} + 0.417179980401234566 + 001y'_n)
 \end{aligned}$$

Corrector

$$\begin{aligned}
 y_{n+1} = y_n + \Delta T & (0.789255401234567893 - 002y'_{n-8} - 0.803895227072310397 - 001y'_{n-7} \\
 & + 0.370351631393298056y'_{n-6} - 0.101879850088183421 + 001y'_{n-5} \\
 & + 0.186150821208112873 + 001y'_{n-4} - 0.238145475088183419 + 001y'_{n-3} \\
 & + 0.220490520282186947 + 001y'_{n-2} - 0.155303461199294532 + 001y'_{n-1} \\
 & + 0.130204433972663139 + 001y'_n + 0.286975446428571432y'_{n+1})
 \end{aligned}$$

10TH-ORDER SYSTEM

Predictor

$$\begin{aligned}
 y_{n+1} = y_n + \Delta T & (0.280189596443936721y'_{n-10} - 0.308887141086793863 + 001y'_{n-9} \\
 & + 0.154861788582752124 + 002y'_{n-8} - 0.466170361852653517 + 002y'_{n-7} \\
 & + 0.936472204560485808 + 002y'_{n-6} - 0.131891420554753887 + 003y'_{n-5} \\
 & + 0.133019135965307840 + 003y'_{n-4} - 0.962690500741542404 + 002y'_{n-3} \\
 & + 0.492504906142275932 + 002y'_{n-2} - 0.172688256657180267 + 002y'_{n-1} \\
 & + 0.445198840045628238 + 001y'_n)
 \end{aligned}$$

Corrector

$$\begin{aligned}
 y_{n+1} = y_n + \Delta T & (-0.678584998463470693 - 002y'_{n-9} + 0.757510538586927482 - 001y'_{n-8} \\
 & - 0.385752772015792851y'_{n-7} + 0.118465362954946289 + 001y'_{n-6} \\
 & - 0.244382699765512266 + 001y'_{n-5} + 0.357154240820907487 + 001y'_{n-4} \\
 & - 0.380648324765512264 + 001y'_{n-3} + 0.301920720097803430 + 001y'_{n-2} \\
 & - 0.185839786130150713 + 001y'_{n-1} + 0.136990283957297846 + 001y'_n \\
 & + 0.280189596443936725y'_{n+1})
 \end{aligned}$$

11TH-ORDER SYSTEM

Predictor

$$\begin{aligned}
 y_{n+1} = y_n + \Delta T & (-0.274265540031599059y'_{n-11} + 0.329711053679152637 + 001y'_{n-10} \\
 & - 0.181734761126058868 + 002y'_{n-9} + 0.607399929634890570 + 002y'_{n-8} \\
 & - 0.137124664395693041 + 003y'_{n-7} + 0.2203578999950647346 + 003y'_{n-6}
 \end{aligned}$$

$$\begin{aligned}
& - 0.258602100049352652 + 003y'_{n-5} + 0.223526764175735529 + 003y'_{n-4} \\
& - 0.141522864179368085 + 003y'_{n-3} + 0.643350953159655414 + 002y'_{n-2} \\
& - 0.202857466060656163 + 002y'_{n-1} + 0.472625394048788144 + 001y'_n)
\end{aligned}$$

Corrector

$$\begin{aligned}
y_{n+1} = y_n + \Delta T & (0.592405641233766230 - 002y'_{n-10} - 0.719504705203489922 - 001y'_{n-9} \\
& + 0.401574156537264175y'_{n-8} - 0.136322208005150713 + 001y'_{n-7} \\
& + 0.313959224562089145 + 001y'_{n-6} - 0.518074106015512264 + 001y'_{n-5} \\
& + 0.630845647070907485 + 001y'_{n-4} - 0.576142186372655120 + 001y'_{n-3} \\
& + 0.399667650901374858 + 001y'_{n-2} - 0.218422096398007856 + 001y'_{n-1} \\
& + 0.143506746010869274 + 001y'_n + 0.274265540031599064y'_{n+1})
\end{aligned}$$

12TH-ORDER SYSTEM

Predictor

$$\begin{aligned}
y_{n+1} = y_n + \Delta T & (0.269028846773648774y'_{n-12} - 0.350261170131538434 + 001y'_{n-11} \\
& + 0.210530144238523454 + 002y'_{n-10} - 0.773598224028086170 + 002y'_{n-9} \\
& + 0.193909272116445200 + 003y'_{n-8} - 0.350195511040422869 + 003y'_{n-7} \\
& + 0.468940554369498813 + 003y'_{n-6} - 0.471672946694082481 + 003y'_{n-5} \\
& + 0.356696043328691671 + 003y'_{n-4} - 0.200709210469570815 + 003y'_{n-3} \\
& + 0.820909992030263604 + 002y'_{n-2} - 0.235140927673494016 + 002y'_{n-1} \\
& + 0.499528278726153021 + 001y'_n)
\end{aligned}$$

$$\begin{aligned}
y_{n+1} = y_n + \Delta T & (-0.523669325795028499 - 002y'_{n-11} + 0.687643755077410822 - 001y'_{n-10} \\
& - 0.417572225545067802y'_{n-9} + 0.155364667328632687 + 001y'_{n-8} \\
& - 0.395538524273689820 + 001y'_{n-7} + 0.728705330591751715 + 001y'_{n-6} \\
& - 0.100194456305011860 + 002y'_{n-5} + 0.104559175310057006 + 002y'_{n-4} \\
& - 0.835358502641194226 + 001y'_{n-3} + 0.514874902576281128 + 001y'_{n-2} \\
& - 0.252984271900479736 + 001y'_{n-1} + 0.149790777920409616 + 001y'_n \\
& + 0.269028846773648779y'_{n+1})
\end{aligned}$$

13TH-ORDER SYSTEM

Predictor

$$\begin{aligned}
y_{n+1} = y_n + \Delta T & (-0.264351348366606509y'_{n-13} + 0.370559637553953339 + 001y'_{n-12} \\
& - 0.241220168739106920 + 002y'_{n-11} + 0.966575000567018070 + 002y'_{n-10} \\
& - 0.266371036484932270 + 003y'_{n-9} + 0.534129457464267776 + 003y'_{n-8} \\
& - 0.803822424837519638 + 003y'_{n-7} + 0.922567468166595582 + 003y'_{n-6} \\
& - 0.811893132041905058 + 003y'_{n-5} + 0.545707257410815325 + 003y'_{n-4} \\
& - 0.276313696102420276 + 003y'_{n-3} + 0.102710404375621668 + 003y'_{n-2} \\
& - 0.269506602961152862 + 002y'_{n-1} + 0.525963413562813672 + 001y'_n)
\end{aligned}$$

Corrector

$$\begin{aligned}
y_{n+1} = y_n + \Delta T & (0.467749840704226446 - 002y'_{n-12} - 0.660441725494997230 - 001y'_{n-11} \\
& + 0.433609251257037710y'_{n-10} - 0.175533676995915544 + 001y'_{n-9}
\end{aligned}$$

$$\begin{aligned}
& + 0.489805803432154596 + 001y'_{n-8} - 0.997532539260029256 + 001y'_{n-7} \\
& + 0.153136405724020429 + 002y'_{n-6} - 0.180460328969857118 + 002y'_{n-5} \\
& + 0.164758579808690949 + 002y'_{n-4} - 0.116979963874471613 + 002y'_{n-3} \\
& + 0.648651357017689891 + 001y'_{n-2} - 0.289468759475409399 + 001y'_{n-1} \\
& + 0.155871525849564560 + 001y'_n + 0.264351348366606514y'_{n+1})
\end{aligned}$$

14TH-ORDER SYSTEM

Predictor

$$\begin{aligned}
y_{n+1} = y_n + \Delta T & (0.260136396127601036y'_{n-14} - 0.390626089415302101 + 001y'_{n-13} \\
& + 0.273780084231512276 + 002y'_{n-12} - 0.118911665064357469 + 003y'_{n-11} \\
& + 0.357054032580430444 + 003y'_{n-10} - 0.787164101532389544 + 003y'_{n-9} \\
& + 0.131531905503545369 + 004y'_{n-8} - 0.169661053634744639 + 004y'_{n-7} \\
& + 0.170375706573778149 + 004y'_{n-6} - 0.133268619708936233 + 004y'_{n-5} \\
& + 0.806103789934543962 + 003y'_{n-4} - 0.371003344292867054 + 003y'_{n-3} \\
& + 0.126382816423233362 + 003y'_{n-2} - 0.305925698419017007 + 002y'_{n-1} \\
& + 0.551977053175573776 + 001y'_n)
\end{aligned}$$

Corrector

$$\begin{aligned}
y_{n+1} = y_n + \Delta T & (-0.421495223900547286 - 002y'_{n-13} + 0.636868297531188845 - 001y'_{n-12} \\
& - 0.449604826298997753y'_{n-11} + 0.196785186625502983 + 001y'_{n-10} \\
& - 0.597450396120363375 + 001y'_{n-9} + 0.133363924168105026 + 002y'_{n-8} \\
& - 0.226328272663337275 + 002y'_{n-7} + 0.297793566566688258 + 002y'_{n-6}
\end{aligned}$$

$$\begin{aligned}
& - 0.307035344707191467 + 002y'_{n-5} + 0.249141923633580516 + 002y'_{n-4} \\
& - 0.159171635786916397 + 002y'_{n-3} + 0.802075618517489103 + 001y'_{n-2} \\
& - 0.327824824850359202 + 001y'_{n-1} + 0.161772458984172222 + 001y'_n \\
& + 0.260136396127601041y'_{n+1})
\end{aligned}$$

15TH-ORDER SYSTEM

Predictor

$$\begin{aligned}
y_{n+1} = y_n + \Delta T & (-0.256309496574389152y'_{n-15} + 0.410477884474343831 + 001y'_{n-14} \\
& - 0.308187580344638819 + 002y'_{n-13} + 0.143998829364498292 + 003y'_{n-12} \\
& - 0.468674127888398661 + 003y'_{n-11} + 0.112675145079332106 + 004y'_{n-10} \\
& - 0.206999313188720724 + 004y'_{n-9} + 0.296467066549164788 + 004y'_{n-8} \\
& - 0.334596214680364058 + 004y'_{n-7} + 0.298658609609259920 + 004y'_{n-6} \\
& - 0.210238361530225295 + 004y'_{n-5} + 0.115596625275858515 + 004y'_{n-4} \\
& - 0.487624165234214118 + 003y'_{n-3} + 0.153295313563544223 + 003y'_{n-2} \\
& - 0.344372122905175380 + 002y'_{n-1} + 0.577608002833012691 + 001y'_n)
\end{aligned}$$

Corrector

$$\begin{aligned}
y_{n+1} = y_n + \Delta T & (0.382689955321188443 - 002y'_{n-14} - 0.616184455371837393 - 001y'_{n-13} \\
& + 0.465511282840366749y'_{n-12} - 0.219084412301040517 + 001y'_{n-11} \\
& + 0.719156975638925207 + 001y'_{n-10} - 0.174666833194989227 + 002y'_{n-9} \\
& + 0.324900246806359841 + 002y'_{n-8} - 0.472589258912522038 + 002y'_{n-7} \\
& + 0.544054552815873020 + 002y'_{n-6} - 0.498571667345446283 + 002y'_{n-5}
\end{aligned}$$

$$\begin{aligned}
& + 0.364063717216533405 + 002y'_{n-4} - 0.211408814688258619 + 002y'_{n-3} \\
& + 0.976199548188629844 + 001y'_{n-2} - 0.368007270159083988 + 001y'_{n-1} \\
& + 0.167512808313990048 + 001y'_n + 0.256309496574389157y'_{n+1})
\end{aligned}$$

16TH-ORDER SYSTEM

Predictor

$$\begin{aligned}
y_{n+1} = y_n + \Delta T & (0.252812146729039235y'_{n-16} - 0.430130384423901690 + 001y'_{n-15} \\
& + 0.344422364522281464 + 002y'_{n-14} - 0.172393560202725853 + 003y'_{n-13} \\
& + 0.601116936411349698 + 003y'_{n-12} - 0.157295758480084203 + 004y'_{n-11} \\
& + 0.315127112179946725 + 004y'_{n-10} - 0.496216409046741608 + 004y'_{n-9} \\
& + 0.621836299289438282 + 004y'_{n-8} - 0.623813310538384942 + 004y'_{n-7} \\
& + 0.501110576709874538 + 004y'_{n-6} - 0.320666707221469633 + 004y'_{n-5} \\
& + 0.161608435980543656 + 004y'_{n-4} - 0.629198967402476089 + 003y'_{n-3} \\
& + 0.183632771171028931 + 003y'_{n-2} - 0.384822066381821657 + 002y'_{n-1} \\
& + 0.602889217505916614 + 001y'_n)
\end{aligned}$$

Corrector

$$\begin{aligned}
y_{n+1} = y_n + \Delta T & (-0.349734984534991753 - 002y'_{n-15} + 0.597844970788105648 - 001y'_{n-14} \\
& - 0.481300426979173843y'_{n-13} + 0.242402719623632056 + 001y'_{n-12} \\
& - 0.855602084154725506 + 001y'_{n-11} + 0.224679938808776918 + 002y'_{n-10} \\
& - 0.454734608810610621 + 002y'_{n-9} + 0.724997069114390405 + 002y'_{n-8} \\
& - 0.922698184009056423 + 002y'_{n-7} + 0.944151375123903584 + 002y'_{n-6}
\end{aligned}$$

$$\begin{aligned}
& - 0.778639442961067678 + 002y'_{n-5} + 0.516827958461417802 + 002y'_{n-4} \\
& - 0.275060581873627117 + 002y'_{n-3} + 0.117205113952822522 + 002y'_{n-2} \\
& - 0.409975468303282999 + 001y'_{n-1} + 0.173108568066549916 + 001y'_n \\
& + 0.252812146729039240y'_{n+1})
\end{aligned}$$

17TH-ORDER SYSTEM

Predictor

$$\begin{aligned}
y_{n+1} = y_n + \Delta T & (-0.249597650297715667y'_{n-17} + 0.449597220179020557 + 001y'_{n-16} \\
& - 0.382465842847283475 + 002y'_{n-15} + 0.204168638654674800 + 003y'_{n-14} \\
& - 0.766435967911289140 + 003y'_{n-13} + 0.214862719645361424 + 004y'_{n-12} \\
& - 0.466197810488537112 + 004y'_{n-11} + 0.800544622478944153 + 004y'_{n-10} \\
& - 0.110298829692048839 + 005y'_{n-9} + 0.122860818726318507 + 005y'_{n-8} \\
& - 0.110923082083738237 + 005y'_{n-7} + 0.810012628718327447 + 004y'_{n-6} \\
& - 0.475117733225696087 + 004y'_{n-5} + 0.221012676751399984 + 004y'_{n-4} \\
& - 0.798925369604922742 + 003y'_{n-3} + 0.217578051611518262 + 003y'_{n-2} \\
& - 0.427253666932433320 + 002y'_{n-1} + 0.627848982535688180 + 001y'_n)
\end{aligned}$$

Corrector

$$\begin{aligned}
y_{n+1} = y_n + \Delta T & (0.321449643132356720 - 002y'_{n-16} - 0.581437891778505560 - 001y'_{n-15} \\
& + 0.496956011738815704y'_{n-14} - 0.266715800027919954 + 001y'_{n-13} \\
& + 0.100745287027864105 + 002y'_{n-12} - 0.284473247585774889 + 002y'_{n-11} \\
& + 0.622506017149331594 + 002y'_{n-10} - 0.107988987477441797 + 003y'_{n-9}
\end{aligned}$$

$$\begin{aligned}
& + 0.150644115156914959 + 003y'_{n-8} - 0.170414226646381561 + 003y'_{n-7} \\
& + 0.156930664108771093 + 003y'_{n-6} - 0.117646552130167235 + 003y'_{n-5} \\
& + 0.715740997631720140 + 002y'_{n-4} - 0.351565596939128017 + 002y'_{n-3} \\
& + 0.139063689685822779 + 002y'_{n-2} - 0.453692619769283512 + 001y'_{n-1} \\
& + 0.178573211999799980 + 001y'_n + 0.249597650297715674y'_{n+1})
\end{aligned}$$

18TH-ORDER SYSTEM

Predictor

$$\begin{aligned}
y_{n+1} = y_n + \Delta T & (0.246628202582257457y'_{n-18} - 0.468890529677834989 + 001y'_{n-17} \\
& + 0.422300871968755965 + 002y'_{n-16} - 0.239495197591850432 + 003y'_{n-15} \\
& + 0.958850938556382618 + 003y'_{n-14} - 0.287954640763607103 + 004y'_{n-13} \\
& + 0.672703314919064167 + 004y'_{n-12} - 0.125106740238631324 + 005y'_{n-11} \\
& + 0.187974031133838633 + 005y'_{n-10} - 0.230209461787542415 + 005y'_{n-9} \\
& + 0.230780387612262725 + 005y'_{n-8} - 0.189410041273515850 + 005y'_{n-7} \\
& + 0.126785322399203019 + 005y'_{n-6} - 0.686428777198174276 + 004y'_{n-5} \\
& + 0.296480906741570766 + 004y'_{n-4} - 0.100017398291204483 + 004y'_{n-3} \\
& + 0.255312166606603652 + 003y'_{n-2} - 0.471646743397239662 + 002y'_{n-1} \\
& + 0.652511802793913926 + 001y'_n)
\end{aligned}$$

Corrector

$$\begin{aligned}
y_{n+1} = y_n + \Delta T & (-0.296944771545820966 - 002y'_{n-17} + 0.566645553095713411 - 001y'_{n-16} \\
& - 0.512469289642956638y'_{n-15} + 0.292002534755271478 + 001y'_{n-14}
\end{aligned}$$

$$\begin{aligned}
& - 0.117536680095813211 + 002y'_{n-13} + 0.355167567288323508 + 002y'_{n-12} \\
& - 0.835721521483436930 + 002y'_{n-11} + 0.156750305811680223 + 003y'_{n-10} \\
& - 0.237926080610462135 + 003y'_{n-9} + 0.295018663082493113 + 003y'_{n-8} \\
& - 0.300351319779401899 + 003y'_{n-7} + 0.251430368205513157 + 003y'_{n-6} \\
& - 0.172771379519933439 + 003y'_{n-5} + 0.970163277892179544 + 002y'_{n-4} \\
& - 0.442430697032149232 + 002y'_{n-3} + 0.163294383043961770 + 002y'_{n-2} \\
& - 0.499125169815794119 + 001y'_{n-1} + 0.183918217887624758 + 001y'_n \\
& + 0.246628202582257464y'_{n+1})
\end{aligned}$$

19TH-ORDER SYSTEM

Predictor

$$\begin{aligned}
y_{n+1} = y_n + \Delta T & (-0.243872812282820742y'_{n-19} + 0.488021163595585154 + 001y'_{n-18} \\
& - 0.463911561971406966 + 002y'_{n-17} + 0.278542842298928894 + 003y'_{n-16} \\
& - 0.118474621800006362 + 004y'_{n-15} + 0.379460399978102219 + 004y'_{n-14} \\
& - 0.949630355049356339 + 004y'_{n-13} + 0.190152964144974132 + 005y'_{n-12} \\
& - 0.309430689218232897 + 005y'_{n-11} + 0.413258857664462777 + 005y'_{n-10} \\
& - 0.455494288318166560 + 005y'_{n-9} + 0.415104336591864297 + 005y'_{n-8} \\
& - 0.312292673926583565 + 005y'_{n-7} + 0.192952893827777942 + 005y'_{n-6} \\
& - 0.970004083320638234 + 004y'_{n-5} + 0.391006008782392086 + 004y'_{n-4} \\
& - 0.123648673801409812 + 004y'_{n-3} + 0.297014417506965999 + 003y'_{n-2} \\
& - 0.517982577730975603 + 002y'_{n-1} + 0.676899084022196000 + 001y'_n)
\end{aligned}$$

Corrector

$$\begin{aligned}
 y_{n+1} = y_n + \Delta T & (0.275539029943671580 - 002y'_{n-18} - 0.553218634047558098 - 001y'_{n-17} \\
 & + 0.527836296513249742y'_{n-16} - 0.318244248979713424 + 001y'_{n-15} \\
 & + 0.135999181481694252 + 002y'_{n-14} - 0.437933464114314523 + 002y'_{n-13} \\
 & + 0.110276006333149324 + 003y'_{n-12} - 0.222410758556360928 + 003y'_{n-11} \\
 & + 0.365008215423706076 + 003y'_{n-10} - 0.492463525691827066 + 003y'_{n-9} \\
 & + 0.549556108163858044 + 003y'_{n-8} - 0.508609229391427752 + 003y'_{n-7} \\
 & + 0.390268974613530393 + 003y'_{n-6} - 0.247530629124250412 + 003y'_{n-5} \\
 & + 0.129056006191068085 + 003y'_{n-4} - 0.549229625038316336 + 002y'_{n-3} \\
 & + 0.189994115045503546 + 002y'_{n-2} - 0.546242343936161959 + 001y'_{n-1} \\
 & + 0.189153459456554518 + 001y'_n + 0.243872812282820749y'_{n+1})
 \end{aligned}$$

20TH-ORDER SYSTEM

Predictor

$$\begin{aligned}
 y_{n+1} = y_n + \Delta T & (0.241305789737813504y'_{n-20} - 0.506998860703909081 + 001y'_{n-19} \\
 & + 0.507283116861404171 + 002y'_{n-18} - 0.321479756498248090 + 003y'_{n-17} \\
 & + 0.144766939357863532 + 004y'_{n-16} - 0.492595118209512418 + 004y'_{n-15} \\
 & + 0.131476164100186736 + 005y'_{n-14} - 0.282023283709688662 + 005y'_{n-13} \\
 & + 0.494125867477697802 + 005y'_{n-12} - 0.714727893661864457 + 005y'_{n-11} \\
 & + 0.859085782552457494 + 005y'_{n-10} - 0.860791492761798119 + 005y'_{n-9} \\
 & + 0.719077239924587968 + 005y'_{n-8} - 0.499352922131336593 + 005y'_{n-7} \\
 & + 0.286483017930154456 + 005y'_{n-6} - 0.134412457973014429 + 005y'_{n-5}
 \end{aligned}$$

$$\begin{aligned}
& + 0.507918663910362727 + 004y'_{n-4} - 0.151157533831520551 + 004y'_{n-3} \\
& + 0.342862517557150564 + 003y'_{n-2} - 0.566243735678538303 + 002y'_{n-1} \\
& + 0.701029662995977350 + 001y'_n)
\end{aligned}$$

Corrector

$$\begin{aligned}
y_{n+1} = y_n + \Delta T & (-0.256702254500723787 - 002y'_{n-19} + 0.540958411995814733 - 001y'_{n-18} \\
& - 0.543056146956131006y'_{n-17} + 0.345424199782150092 + 001y'_{n-16} \\
& - 0.156196667203572017 + 002y'_{n-15} + 0.533990356859616411 + 002y'_{n-14} \\
& - 0.143291140255911992 + 003y'_{n-13} + 0.309271594022110403 + 003y'_{n-12} \\
& - 0.545778588550922682 + 003y'_{n-11} + 0.796165322083121750 + 003y'_{n-10} \\
& - 0.966736343017184307 + 003y'_{n-9} + 0.980713214823273717 + 003y'_{n-8} \\
& - 0.831977059385989507 + 003y'_{n-7} + 0.589264562302491472 + 003y'_{n-6} \\
& - 0.347028422968730952 + 003y'_{n-5} + 0.168855123728860301 + 003y'_{n-4} \\
& - 0.673601867343917011 + 002y'_{n-3} + 0.219258172058586058 + 002y'_{n-2} \\
& - 0.595015772291299479 + 001y'_{n-1} + 0.194287504546568993 + 001y'_n \\
& + 0.241305789737813511y'_{n+1})
\end{aligned}$$

APPENDIX C

ADAMS PREDICTOR AND CORRECTOR DIFFERENCE FORMULAS

The Adams predictor and corrector difference formulas are expressed as follows.

PREDICTOR DIFFERENCE FORMULA

$$\begin{aligned}
 y_{n+1} = & y_n + \Delta T (y'_n + 0.5 \Delta_1 y'_n + 0.4166666666666667 \Delta_2 y'_n + 0.375 \Delta_3 y'_n \\
 & + 0.3486111111111111 \Delta_4 y'_n + 0.3298611111111111 \Delta_5 y'_n \\
 & + 0.315591931216931216 \Delta_6 y'_n + 0.304224537037037036 \Delta_7 y'_n \\
 & + 0.294868000440917107 \Delta_8 y'_n + 0.286975446428571428 \Delta_9 y'_n \\
 & + 0.280189596443936721 \Delta_{10} y'_n + 0.274265540031599059 \Delta_{11} y'_n \\
 & + 0.269028846773648774 \Delta_{12} y'_n + 0.264351348366606509 \Delta_{13} y'_n \\
 & + 0.260136396127601036 \Delta_{14} y'_n + 0.256309496574389152 \Delta_{15} y'_n \\
 & + 0.252812146729039235 \Delta_{16} y'_n + 0.249597650297715667 \Delta_{17} y'_n \\
 & + 0.246628202582257457 \Delta_{18} y'_n + 0.243872812282820742 \Delta_{19} y'_n \\
 & + 0.241305789737813504 \Delta_{20} y'_n)
 \end{aligned}$$

CORRECTOR DIFFERENCE FORMULA

$$\begin{aligned}
 y_{n+1} = & y_n + \Delta T (y'_{n+1} - 0.5 \Delta_1 y'_{n+1} - 0.8333333333333335 - 001 \Delta_2 y'_{n+1} \\
 & - 0.41666666666666665 - 001 \Delta_3 y'_{n+1} - 0.26388888888888892 - 001 \Delta_4 y'_{n+1} \\
 & - 0.1875 - 001 \Delta_5 y'_{n+1} - 0.142691798941798941 - 001 \Delta_6 y'_{n+1} \\
 & - 0.113673941798941799 - 001 \Delta_7 y'_{n+1} - 0.935653659611992939 - 002 \Delta_8 y'_{n+1}
 \end{aligned}$$

$$\begin{aligned}
& - 0.789255401234567893 - 002 \Delta_9 y'_{n+1} - 0.678584998463470693 - 002 \Delta_{10} y'_{n+1} \\
& - 0.592405641233766230 - 002 \Delta_{11} y'_{n+1} - 0.523669325795028499 - 002 \Delta_{12} y'_{n+1} \\
& - 0.467749840704226446 - 002 \Delta_{13} y'_{n+1} - 0.421495223900547286 - 002 \Delta_{14} y'_{n+1} \\
& - 0.382689955321188443 - 002 \Delta_{15} y'_{n+1} - 0.349734984534991753 - 002 \Delta_{16} y'_{n+1} \\
& - 0.321449643132356720 - 002 \Delta_{17} y'_{n+1} - 0.296944771545820966 - 002 \Delta_{18} y'_{n+1} \\
& - 0.275539029943671580 - 002 \Delta_{19} y'_{n+1} - 0.256702254500723787 - 002 \Delta_{20} y'_{n+1})
\end{aligned}$$

APPENDIX D

EXPANSION OF DIFFERENCE FORMULAS IN TERMS OF TABULATED VALUES

The expansion of the backward difference formula is as follows .

$$\Delta_1 y'_n = y'_n - y'_{n-1}$$

$$\Delta_2 y'_n = y'_n - 2y'_{n-1} + y'_{n-2}$$

$$\Delta_3 y'_n = y'_n - 3y'_{n-1} + 3y'_{n-2} - y'_{n-3}$$

$$\Delta_4 y'_n = y'_n - 4y'_{n-1} + 6y'_{n-2} - 4y'_{n-3} + y'_{n-4}$$

$$\Delta_5 y'_n = y'_n - 5y'_{n-1} + 10y'_{n-2} - 10y'_{n-3} + 5y'_{n-4} - y'_{n-5}$$

$$\Delta_6 y'_n = y'_n - 6y'_{n-1} + 15y'_{n-2} - 20y'_{n-3} + 15y'_{n-4} - 6y'_{n-5} + y'_{n-6}$$

$$\Delta_7 y'_n = y'_n - 7y'_{n-1} + 21y'_{n-2} - 35y'_{n-3} + 35y'_{n-4} - 21y'_{n-5} + 7y'_{n-6} - y'_{n-7}$$

$$\Delta_8 y'_n = y'_n - 8y'_{n-1} + 28y'_{n-2} - 56y'_{n-3} + 70y'_{n-4} - 56y'_{n-5} + 28y'_{n-6} - 8y'_{n-7} + y'_{n-8}$$

$$\Delta_9 y'_n = y'_n - 9y'_{n-1} + 36y'_{n-2} - 84y'_{n-3} + 126y'_{n-4} - 126y'_{n-5} + 84y'_{n-6} - 36y'_{n-7} + 9y'_{n-8} - y'_{n-9}$$

$$\Delta_{10} y'_n = y'_n - 10y'_{n-1} + 45y'_{n-2} - 120y'_{n-3} + 210y'_{n-4} - 252y'_{n-5} + 210y'_{n-6} - 120y'_{n-7} + 45y'_{n-8} - 10y'_{n-9} + y'_{n-10}$$

$$\Delta_{11}y'_n = y'_n - 11y'_{n-1} + 55y'_{n-2} - 165y'_{n-3} + 330y'_{n-4} - 462y'_{n-5} + 462y'_{n-6} - 330y'_{n-7} \\ + 165y'_{n-8} - 55y'_{n-9} + 11y'_{n-10} - y'_{n-11}$$

$$\Delta_{12}y'_n = y'_n - 12y'_{n-1} + 66y'_{n-2} - 220y'_{n-3} + 495y'_{n-4} - 792y'_{n-5} + 924y'_{n-6} - 792y'_{n-7} \\ + 495y'_{n-8} - 220y'_{n-9} + 66y'_{n-10} - 12y'_{n-11} + y'_{n-12}$$

$$\Delta_{13}y'_n = y'_n - 13y'_{n-1} + 78y'_{n-2} - 286y'_{n-3} + 715y'_{n-4} - 1287y'_{n-5} + 1716y'_{n-6} - 1716y'_{n-7} \\ + 1287y'_{n-8} - 715y'_{n-9} + 286y'_{n-10} - 78y'_{n-11} + 13y'_{n-12} - y'_{n-13}$$

$$\Delta_{14}y'_n = y'_n - 14y'_{n-1} + 91y'_{n-2} - 364y'_{n-3} + 1001y'_{n-4} - 2002y'_{n-5} + 3003y'_{n-6} - 3432y'_{n-7} \\ + 3003y'_{n-8} - 2002y'_{n-9} + 1001y'_{n-10} - 364y'_{n-11} + 91y'_{n-12} - 14y'_{n-13} + y'_{n-14}$$

$$\Delta_{15}y'_n = y'_n - 15y'_{n-1} + 105y'_{n-2} - 455y'_{n-3} + 1365y'_{n-4} - 3003y'_{n-5} + 5005y'_{n-6} - 6435y'_{n-7} \\ + 6435y'_{n-8} - 5005y'_{n-9} + 3003y'_{n-10} - 1365y'_{n-11} + 455y'_{n-12} - 105y'_{n-13} + 15y'_{n-14} \\ - y'_{n-15}$$

$$\Delta_{16}y'_n = y'_n - 16y'_{n-1} + 120y'_{n-2} - 560y'_{n-3} + 1820y'_{n-4} - 4368y'_{n-5} + 8008y'_{n-6} - 11440y'_{n-7} \\ + 12870y'_{n-8} - 11440y'_{n-9} + 8008y'_{n-10} - 4368y'_{n-11} + 1820y'_{n-12} - 560y'_{n-13} \\ + 120y'_{n-14} - 16y'_{n-15} + y'_{n-16}$$

$$\Delta_{17}y'_n = y'_n - 17y'_{n-1} + 136y'_{n-2} - 680y'_{n-3} + 2380y'_{n-4} - 6188y'_{n-5} + 12376y'_{n-6} - 19448y'_{n-7} \\ + 24310y'_{n-8} - 24310y'_{n-9} + 19448y'_{n-10} - 12376y'_{n-11} + 6188y'_{n-12} - 2380y'_{n-13} \\ + 680y'_{n-14} - 136y'_{n-15} + 17y'_{n-16} - y'_{n-17}$$

$$\begin{aligned}\Delta_{18}y'_n = & y'_n - 18y'_{n-1} + 153y'_{n-2} - 816y'_{n-3} + 3060y'_{n-4} - 8568y'_{n-5} + 18564y'_{n-6} \\ & - 31824y'_{n-7} + 43758y'_{n-8} - 48620y'_{n-9} + 43758y'_{n-10} - 31824y'_{n-11} + 18564y'_{n-12} \\ & - 8568y'_{n-13} + 3060y'_{n-14} - 816y'_{n-15} + 153y'_{n-16} - 18y'_{n-17} + y'_{n-18}\end{aligned}$$

$$\begin{aligned}\Delta_{19}y'_n = & y'_n - 19y'_{n-1} + 171y'_{n-2} - 969y'_{n-3} + 3876y'_{n-4} - 11628y'_{n-5} + 27132y'_{n-6} - 50388y'_{n-7} \\ & + 75582y'_{n-8} - 92378y'_{n-9} + 92378y'_{n-10} - 75582y'_{n-11} + 50388y'_{n-12} - 27132y'_{n-13} \\ & + 11628y'_{n-14} - 3876y'_{n-15} + 969y'_{n-16} - 171y'_{n-17} + 19y'_{n-18} - y'_{n-19}\end{aligned}$$

$$\begin{aligned}\Delta_{20}y'_n = & y'_n - 20y'_{n-1} + 190y'_{n-2} - 1140y'_{n-3} + 4845y'_{n-4} - 15504y'_{n-5} + 38760y'_{n-6} \\ & - 77520y'_{n-7} + 125970y'_{n-8} - 167960y'_{n-9} + 184756y'_{n-10} - 167960y'_{n-11} \\ & + 125970y'_{n-12} - 77520y'_{n-13} + 38760y'_{n-14} - 15504y'_{n-15} + 4845y'_{n-16} \\ & - 1140y'_{n-17} + 190y'_{n-18} - 20y'_{n-19} + y'_{n-20}\end{aligned}$$